Model answers final exam
Quantum Physics 1 – 30 Oct. 2014

Problem 1 a) $\langle \hat V \rangle = \int_{-\infty}^{\infty} \Psi^*_{(x)} V(x)\, \Psi(x)\, dx$

$= \int_{x_0}^{x_4} -iAx\, V(x)\, iAx\, dx = A^2 \int_{x_0}^{x_4} x^2 V(x)\, dx$, with

$x_0 = 0$ nm, $x_1 = 1$ nm etc and $V_0 = 0$ eV, $V_1 = 1$ eV etc.,

and A^2 from normalization.

$$\left.\int_{x_0}^{x_4} A^2 x^2\, dx = 1 \Rightarrow A^2\left[\tfrac{1}{3}x^3\right]_{x_0}^{x_4} = 1 \Rightarrow A^2 = \frac{3}{x_4^3} = \frac{3}{(4x_1)^3}\right\} \Rightarrow$$

$$\langle \hat V \rangle = \int_{x_1}^{x_3} A^2 x^2 V_4\, dx + \int_{x_3}^{x_4} A^2 x^2 V_1\, dx = A^2\left(\left[V_4 \frac{x^3}{3}\right]_{x_1}^{x_3} + \left[V_1 \frac{x^3}{3}\right]_{x_3}^{x_4}\right)$$

$$= \frac{3}{(4x_1)^3}\left(4V_1\left(\frac{27x_1^3}{3} - \frac{x_1^3}{3}\right) + V_1\left(\frac{64x_1^3}{3} - \frac{27x_1^3}{3}\right)\right) = V_1\left(\frac{3}{64}\left(\frac{4\cdot 26}{3} + \frac{37}{3}\right)\right) = \frac{141}{64}\ \underline{\underline{\text{eV}}}$$ unit

b) $\langle \hat T \rangle = \int_{-\infty}^{\infty} \Phi^*(k)\, \frac{-\hbar^2 k^2}{2m} \Phi(k)\, dk$, use $k_1 = 1\,\text{nm}^{-1}$, $k_2 = 2\,\text{nm}^{-1}$, $k_3 = 3\,\text{nm}^{-1}$

$$\left.\Rightarrow \langle \hat T \rangle = +\frac{\hbar^2}{2m}\int_{k_1}^{k_3} B^2 k^2\, dk = +\frac{\hbar^2}{2m} B^2 \left[\tfrac{1}{3}k^3\right]_{k_1}^{3k_1} = +\frac{\hbar^2}{2m} B^2\left(\frac{26}{3}k_1^3\right)\right\} \Rightarrow$$

Normalization gives $B^2 \cdot 2k_1 = 1 \Rightarrow B^2 = \frac{1}{2k_1}$

$\langle \hat T \rangle = \frac{\hbar^2}{m}\frac{26}{12}k_1^2$. With $m = 9.1\cdot 10^{-31}$ kg this gives

$$\langle \hat T \rangle = \frac{(1.055\cdot 10^{-34})^2}{9.1\cdot 10^{-31}} \cdot \frac{26}{12} \cdot (10^9)^2 = 2.65\cdot 10^{-20}\ \text{J}$$ ← unit!

c) Calculate $\Psi(x)$ via Fourier transform to get information about position (or calculate $\langle \hat x \rangle = \int_{-\infty}^{\infty} \Phi(k)^* \hat x\, \Phi(k)\, dk = 0$ and $\hat x \leftrightarrow i\frac{\partial}{\partial k}$)

Since $\Phi(k)$ is a block function, $\Psi(x)$ must be in the form of a sinc function.

$$\Psi(x) = \frac{1}{\sqrt{2\pi}}\int_{k_1}^{3k_1} e^{ikx} B\, dk = \frac{B}{\sqrt{2\pi}}\left[\frac{1}{ix}e^{ikx}\right]_{k_1}^{3k_1}$$

$$= \frac{B}{\sqrt{2\pi}}\left[\frac{1}{ix}\left(e^{i3k_1x} - e^{ik_1x}\right)\right] = \frac{B}{\sqrt{2\pi}}\frac{1}{ix}\left(e^{i(2k_1+k_1)x} - e^{i(2k_1-k_1)x}\right)$$

$$= \frac{B}{\sqrt{2\pi}} e^{i2k_1x}\frac{2ik_1}{ixk_1}\frac{e^{ik_1x} - e^{-ik_1x}}{2i} = \frac{2B}{\sqrt{2\pi}} e^{2ik_1x}\frac{\sin(k_1x)}{k_1x} \Rightarrow$$

$\Psi^*(x)\Psi(x) = \frac{2B^2}{\pi}\left(\frac{\sin(k_1x)}{k_1x}\right)^2 \Rightarrow$ is a sinc function

centered at $x = 0$ with a main peak of width $\Delta x \approx \frac{\frac{1}{2}\pi}{k_1}$

$\Rightarrow \langle x \rangle = 0$.

d) Using $\Delta x \approx \frac{\pi}{2k_1}$ from c), or using $\Delta x \Delta p = \Delta x\, \hbar \Delta k \gtrsim \frac{\hbar}{2}$
$\Rightarrow \Delta x \approx \frac{1}{2k_1}$ from Heisenberg, we take as a starting point $\Delta x \approx \frac{1}{k_1}$. The question is therefore, how long does it take for Δx to grow to $\Delta x \approx \frac{100}{k_1}$? The width increases roughly as $\Delta x(t) = \Delta x(t{=}0) + (v_{max} - v_{min})\, t \Rightarrow$

$$\Delta x(t) \approx (v_{max} - v_{min})\, t \Rightarrow t \approx \frac{100/k_1}{v_{max} - v_{min}}$$

$$\xRightarrow{v = \frac{\hbar k}{m}} t \approx \frac{m \cdot 100/k_1}{\hbar(k_{max} - k_{min})} \approx \frac{100\, m}{2\hbar k_1^2}$$

Filling in m, $\hbar$, k_1 gives $t \approx 0.4\cdot 10^{-12}$ s unit!

(3/8)

Problem 2 a) $E_g = \frac{\pi^2 \hbar^2}{2mL^2}$

$E_e = \frac{4\pi^2 \hbar^2}{2mL^2}$

b) $\Psi(x,t=0) = \frac{1}{\sqrt{2}}(\varphi_g + \varphi_e) \Rightarrow$ is $\frac{1}{\sqrt{2}}$ (" " + " ")

After a time t_R this state has evolved to one that gives $|\Psi(x)|^2$ of the right plot ⇒

$\Psi(x, t=t_R)$ is as $\frac{1}{\sqrt{2}}$ (" " + " ") $= \frac{1}{\sqrt{2}}$ (" " − " ")

$\Rightarrow \Psi(x,t=t_R) = e^{i\theta} \cdot \frac{1}{\sqrt{2}}(\varphi_g - \varphi_e)$, where θ is a global phase for the state.

In general $\Psi(x,t) = \hat{U}\Psi(x,0) = \frac{1}{\sqrt{2}}\left(e^{-\frac{iE_g t}{\hbar}}\varphi_g + e^{-\frac{iE_e t}{\hbar}}\varphi_e\right)$

$= \frac{1}{\sqrt{2}} e^{-\frac{iE_g t}{\hbar}}\left(\varphi_g + e^{-i\frac{(E_e-E_g)t}{\hbar}}\varphi_e\right) \Rightarrow$ use $e^{-i\pi} = -1$

$(E_e - E_g)t_R/\hbar = \pi \Rightarrow t_R = \frac{\pi\hbar}{(E_e - E_g)}$

c) $\langle\hat{x}\rangle(t) = \int_{-\infty}^{\infty} \Psi(x,0)^* \hat{U}^\dagger \hat{x} \hat{U} \Psi(x,0)\, dx$ Use $\omega_g = E_g/\hbar$, $\omega_e = E_e/\hbar$

$= \frac{1}{2}\int_{-\infty}^{\infty} \left(e^{+i\omega_g t}\varphi_g^* + e^{+i\omega_e t}\varphi_e^*\right) x \left(e^{-i\omega_g t}\varphi_g + e^{-i\omega_e t}\varphi_e\right) dx$

$= \frac{1}{2}\int_{-\infty}^{\infty} \varphi_g^2 x + \varphi_e^2 x + 2\cos\left((\omega_e - \omega_g)t\right)\varphi_g\varphi_e x \; dx$.

Calculate first

$\int_{-\infty}^{\infty} \varphi_g^2 x\, dx = \int_0^L \frac{2}{L}\sin^2\left(\frac{\pi}{L}x\right) x\, dx = \int_0^1 \frac{2}{L}\sin^2\left(\frac{\pi}{L}x\right)\left(\frac{x}{L}\right) L^2\, d\left(\frac{x}{L}\right)$

$= 2L \cdot \frac{1}{4} = \frac{1}{2}L$

(4/8)

$\int_{-\infty}^{\infty} \varphi_e^2 x\, dx = \int_0^L \frac{2}{L}\sin^2\left(2\pi\frac{x}{L}\right) x\, dx$

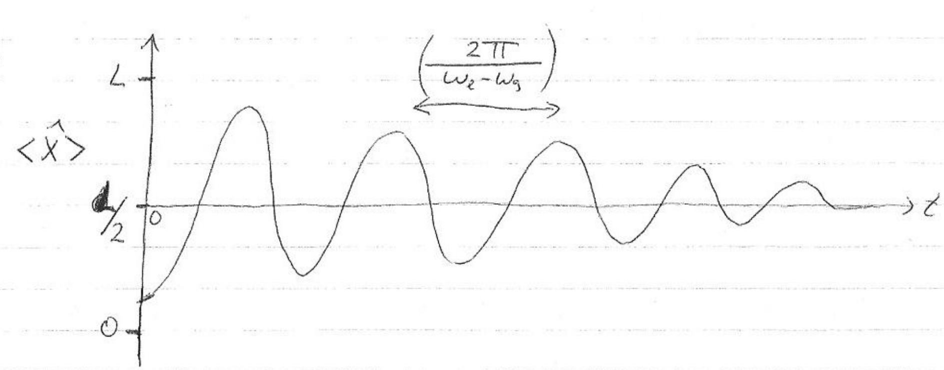
$= \int_0^1 \frac{2L^2}{L}\sin^2\left(2\pi\frac{x}{L}\right)\left(\frac{x}{L}\right) d\left(\frac{x}{L}\right) = 2L \cdot \frac{1}{4} = \frac{1}{2}L$

$\int_{-\infty}^{\infty} \varphi_g\varphi_e x\, dx = \int_0^L \frac{2}{L}\sin\left(\pi\frac{x}{L}\right)\sin\left(2\pi\frac{x}{L}\right) x\, dx$

$= \int_0^1 \frac{2L^2}{L}\sin\left(\pi\frac{x}{L}\right)\sin\left(2\pi\frac{x}{L}\right)\left(\frac{x}{L}\right) d\left(\frac{x}{L}\right) = -2L\frac{8}{9\pi^2} = -\frac{16L}{9\pi^2}$ ⇒

$\langle\hat{x}\rangle(t) = \frac{1}{2}L - \underbrace{\frac{32}{9\pi^2}}_{0.36}\cos\left((\omega_e - \omega_g)t\right) L$

d)



Due to the weak coupling to the cold environment the system will decay to the ground state, which has $\langle\hat{x}\rangle = L/2$. Before fully relaxing it will show some damped oscillations of $\langle\hat{x}\rangle(t)$, as calculated for question c).

5/8

Problem 3

a) $\langle \psi_1 | \hat{S}_y | \psi_1 \rangle = (a^* \, b^*) \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} \frac{\hbar}{2} \overset{a^*=a,\ b^*=b}{=}$

$= (a \ b) \begin{pmatrix} -ib \\ +ia \end{pmatrix} \frac{\hbar}{2} = \frac{\hbar}{2}(-iab + iab) = 0 \, \hbar.$

b) $(\Delta S_z)^2 = \langle S_z^2 \rangle - \langle S_z \rangle^2$

$\hat{S}_z \leftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \frac{\hbar}{2} \Rightarrow \hat{S}_z^2 \leftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \frac{\hbar^2}{4}$

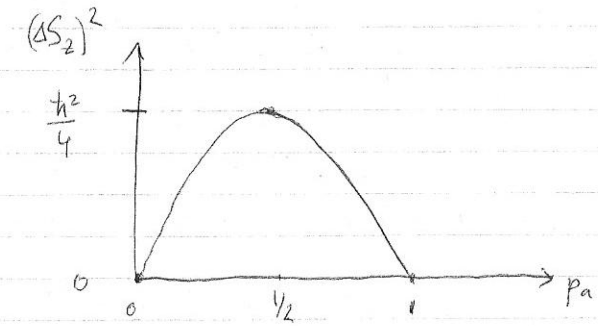
$\langle \psi_1 | S_z | \psi_1 \rangle = (a \ b) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} \frac{\hbar}{2} = (a^2 - b^2) \frac{\hbar}{2}$

$\langle \psi_1 | S_z^2 | \psi_1 \rangle = (a \ b) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} \frac{\hbar^2}{4} = (a^2 + b^2) \frac{\hbar^2}{4}$ $\Rightarrow$

Use $p_a = a^2$ and $b^2 = 1 - a^2 = (1 - p_a)$ since $a^2 + b^2 = 1$

$(\Delta S_z)^2 = \frac{\hbar^2}{4} - \frac{\hbar^2}{4}\left(p_a - (1-p_a)\right)^2 = \hbar^2\left(\frac{1}{4} - \frac{1}{4}(2p_a - 1)^2\right)$

$= \hbar^2 p_a (1 - p_a) = \hbar^2 (p_a - p_a^2)$



6/8

c) $(\Delta S_x)^2 = \langle S_x^2 \rangle - \langle S_x \rangle^2$

$\hat{S}_x \leftrightarrow \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{\hbar}{2} \Rightarrow \hat{S}_x^2 \leftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \frac{\hbar^2}{4}$

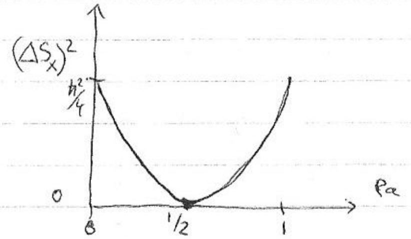
$\langle \psi_1 | \hat{S}_x | \psi_1 \rangle = (a \ b) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} \frac{\hbar}{2} = (2ab) \frac{\hbar}{2}$

$\langle \psi_1 | S_x^2 | \psi_1 \rangle = (a \ b) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} \frac{\hbar^2}{4} = (a^2 + b^2) \frac{\hbar^2}{4} = \frac{\hbar^2}{4}$ $\Rightarrow$

$(\Delta S_x)^2 = \frac{\hbar^2}{4} - \frac{\hbar^2}{4}(4a^2b^2) = \frac{\hbar^2}{4}\left(1 - 4p_a(1-p_a)\right)$

$= \hbar^2\left(\frac{1}{4} - p_a(1-p_a)\right)$

$= \hbar^2\left(\frac{1}{4} - p_a + p_a^2\right)$



d) $\Delta S_x = 0$ for $p_a = \frac{1}{2} \Rightarrow a = b = \frac{1}{\sqrt{2}}$

Here the state is an eigenstate of $\hat{S}_x$, here $|\psi_1\rangle = \frac{|\uparrow\rangle + |\downarrow\rangle}{\sqrt{2}}$ $= |\uparrow_x\rangle$ which fulfills $\hat{S}_x |\uparrow_x\rangle = \frac{\hbar}{2} |\uparrow_x\rangle \Rightarrow |\uparrow_x\rangle$ has $\Delta S_x = 0$.

e) $\Delta S_x \cdot \Delta S_z = 0$ for $p_a = 0$, $\frac{1}{2}$ and 1.

The generalized uncertainty principle gives

$(\Delta S_z)^2 (\Delta S_x)^2 \geq \left(\frac{1}{2i} \langle [\hat{S}_z, \hat{S}_x] \rangle\right)^2$ and $[\hat{S}_z, \hat{S}_x] = i\hbar \hat{S}_y$

$\Rightarrow (\Delta S_z)^2 (\Delta S_x)^2 \geq \left(\frac{\hbar}{2} \langle \hat{S}_y \rangle\right)^2.$

Since for $|\psi_1\rangle$ the value of $\langle \hat{S}_y \rangle = 0$ (see a)),

it is allowed to have $\Delta S_x \cdot \Delta S_y = 0$.

Problem 4 a)

The rules for addition of angular momentum give

$$\ell_{tot} = |\ell_2 - \ell_1|,\ |\ell_2 - \ell_1|+1,\ \ldots\ |\ell_2+\ell_1|-1,\ |\ell_2+\ell_1|$$

and the values for $|\vec{L}_{tot}|$ are then $\sqrt{\ell_{tot}(\ell_{tot}+1)}\,\hbar \Rightarrow$

Here $\ell_{tot} = \overset{9/2-5/2}{2},\ 3,\ 4,\ 5,\ 6,\ \overset{9/2+5/2}{7} \Rightarrow$

The possible measurement outcomes are

$$\sqrt{6}\,\hbar,\ \sqrt{12}\,\hbar,\ \sqrt{20}\,\hbar,\ \sqrt{30}\,\hbar,\ \sqrt{42}\,\hbar,\ \sqrt{56}\,\hbar.$$

b) The lowest value was measured, so the state after the measurement has $\ell_{tot} = 2$. The possible outcomes for $\hat{L}_{TOT,y}$ are governed by

$$\hat{L}_{TOT,y}\,|\ell_{tot}, m_y\rangle = \hbar m_y\,|\ell_{tot}, m_y\rangle \text{ with } m_y = -\ell_{tot}, -(\ell_{tot}-1) \cdots + \ell_{tot}$$

$\Rightarrow$ The possible measurement outcomes are

$$-2\hbar,\ -1\hbar,\ 0\hbar,\ +1\hbar,\ +2\hbar$$

c) A system with $\ell = 1$ has for its operator $\hat{L}_x$ the eigen value equation

$$\hat{L}_x\,|\ell, m_x\rangle = m_x \hbar\,|\ell, m_x\rangle \text{ with } m_x = -1, 0, 1.$$

So, the eigen values are $-\hbar$, $0\hbar$, and $+\hbar$

and these define the possible measurement outcomes when $\vec{L}_x$ is measured.

d) The probabilities are

$$P_i = |\langle \Psi_2 | \Psi_m \rangle|^2$$

$\uparrow$ +, 0, or − $\qquad$ $\uparrow$ $|+_x\rangle$, $|0_x\rangle$ or $|-_x\rangle$. $\Rightarrow$

$$P_+ = |\langle \Psi_2 | +_x \rangle|^2 = \left| \left(\sqrt{\tfrac{1}{5}}, 0, \sqrt{\tfrac{4}{5}}\right) \begin{pmatrix} \frac{1}{2} \\ \sqrt{\frac{1}{2}} \\ \frac{1}{2} \end{pmatrix} \right|^2 = \left| \sqrt{\tfrac{1}{20}} + 0 + \sqrt{\tfrac{4}{20}} \right|^2 = 9/20$$

$$P_0 = |\langle \Psi_2 | 0_x \rangle|^2 = \left| \left(\sqrt{\tfrac{1}{5}}, 0, \sqrt{\tfrac{4}{5}}\right) \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{pmatrix} \right|^2 = \left| \sqrt{\tfrac{1}{10}} + 0 - \sqrt{\tfrac{4}{10}} \right|^2 = 2/20 = 1/10$$

$$P_- = |\langle \Psi_2 | -_x \rangle|^2 = \left| \left(\sqrt{\tfrac{1}{5}}, 0, \sqrt{\tfrac{4}{5}}\right) \begin{pmatrix} \sqrt{\frac{1}{4}} \\ -1/\sqrt{2} \\ \sqrt{\frac{1}{4}} \end{pmatrix} \right|^2 = \left| \sqrt{\tfrac{1}{20}} + 0 + \sqrt{\tfrac{4}{20}} \right|^2 = 9/20$$

$$\overline{\qquad\qquad} +$$

$$1$$